Zero-Sum Flows for Steiner Triple Systems

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Abstract

Given a 2-(v, k, λ) design, $\mathcal{S} = (X, \mathcal{B})$, a *zero-sum n-flow* of $\mathcal{S}$ is a map $f : \mathcal{B} \longrightarrow \{\pm 1, \ldots, \pm(n-1)\}$ such that for any point $x \in X$, the sum of f around all the blocks incident with x is zero. It has been conjectured that every Steiner triple system, STS(v), on v points $(v > 7)$ admits a zero-sum 3-flow. We show that for every pair (v, λ), for which a triple system, TS(v, λ) exists, there exists one which has a zero-sum 3-flow, except when $(v, \lambda) \in \{(3, 1), (4, 2), (6, 2), (7, 1)\}$ and except possibly when $v \equiv 10 \pmod{12}$ and $\lambda = 2$. We also give a $O(\lambda^2 v^2)$ bound on n and a recursive result which shows that every STS(v) with a zero-sum 3-flow can be embedded in an STS$(2v + 1)$ with a zero-sum 3-flow if $v \equiv 3 \pmod 4$, a zero-sum 4-flow if $v \equiv 3 \pmod 6$ and with a zero-sum 5-flow if $v \equiv 1 \pmod 4$.

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1 Introduction

Let G be a graph with the vertex set $V(G)$ and edge set $E(G)$. A *k-edge colouring* of a graph G is a function $f : E(G) \longrightarrow L$ such that $|L| = k$ and

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$f(e_1) \neq f(e_2)$ for every two adjacent edges e_1 and e_2. The *chromatic index* of G, denoted by $\chi'(G)$ is the minimum number k for which G has a k-edge colouring. A 1-*factor* of a graph G is a set of independent edges which covers all vertices of G. Given a colouring of G, a *rainbow* 1-*factor* is a 1-factor all of whose edges have different colours. We denote the complete graph of order n by K_n.

A *zero-sum flow* of a graph G is an assignment of non-zero real numbers to the edges of G such that the sum of the values of all edges incident with any given vertex is zero. Let n be a natural number. A *zero-sum n-flow* is a zero-sum flow with values from the set $\{\pm 1, \ldots, \pm(n-1)\}$. For a subset $S \subseteq E(G)$, the *weight of S* is defined to be the sum of the values of all edges of S. Such flows have been studied in [1, 2, 4, 5, 29]. Zero-sum flows are motivated by nowhere-zero flows, which were first introduced by Tutte in 1949 [27].

Let n be a positive integer. A *Latin square of order* n with entries from X is an $n \times n$ array L such that every row and column of L is a permutation of X. Suppose that L_1 is a Latin square of order n with entries from X and L_2 is a Latin square of order n with entries from Y. We say that L_1 and L_2 are *orthogonal* provided that, for every $x \in X$ and for every $y \in Y$, there is a unique cell (i, j) such that $L_1(i, j) = x$ and $L_2(i, j) = y$. A Latin square, L, is called *indempotent* if $L(i, i) = i$ for every i. It is well known that for every positive integer $n \neq 2, 6$, there exist two orthogonal Latin squares of order n. A *transversal* of a Latin square is a set of cells which between them contain each row, column and entry exactly once. If a square L has an orthogonal mate, L', it is possible to partition the cells of L into transversals, T_i, each of which correspond to the positions of the entry i in L'. We refer the reader to [16] for notation and further results on Latin squares.

A 2-(v, k, λ) *design* $\mathcal{D}$ (briefly, 2-design), is a pair $(X, \mathcal{B})$, where X is a v-set of points and $\mathcal{B}$ is a collection of k-subsets of X, called *blocks*, with the property that every 2-subset of X is contained in exactly λ blocks. Traditionally, the number of blocks and the frequency of occurrences of points in blocks are denoted by b and r, respectively. Given an indexing of the points and blocks of a 2-design, the *incidence matrix* of $\mathcal{D}$ is the $v \times b$ $(0, 1)$-matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & x_i \in B_j,\ B_j \in \mathcal{B} \\ 0 & \text{otherwise.} \end{cases}$$

We refer the reader to [16] for notation and further results on designs.

Given a 2-design $\mathcal{D} = (X, \mathcal{B})$, we define a zero-sum n-flow of $\mathcal{D}$ to be a function $f : \mathcal{B} \longrightarrow \{\pm 1, \ldots, \pm(n-1)\}$ such that the sum of the block weights around any point is zero, i.e.

$$w(x) = \sum_{x \in B} f(B) = 0.$$

This is equivalent to finding a vector in the nullspace of the incidence matrix of the design whose entries are all in the set $\{\pm 1, \ldots, \pm(n-1)\}$. Zero-sum n-flow of designs have been previously been studied in [3].

A *parallel class* of a 2-design is a collection of disjoint blocks which between them contain every point of X exactly once. A 2-design is called *resolvable* if there exists a partition of the set of blocks into parallel classes. An α*-resolution class* is a collection of blocks $\mathcal{S} \subseteq \mathcal{B}$ which contain every point of X exactly α times. If the block set $\mathcal{B}$ can be partitioned into α-resolution classes we call the design α*-resolvable*; in this case we denote the number of α-resolution classes by $\rho = r/\alpha$.

An *automorphism* of a 2-design is a permutation of the point set which maps blocks to blocks. The set of automorphisms of a 2-design forms a group, called the *automorphism group*, G, of the design. A 2-(v, k, λ) design, $(X, \mathcal{B})$, is called *cyclic* if its automorphism group contains a cycle of length v. In such cases there exists a set of *starter blocks* $\mathcal{S} \subseteq \mathcal{B}$, such that the full design may be obtained by acting on these blocks by G. The orbit of a starter block which has length less than v is called a *short orbit*; if the orbit is of length v, it is called a *full orbit*.

A 2-$(v, 3, 1)$ design is called a *Steiner triple system* of order v, denoted STS(v), and a resolvable STS(v) is called a *Kirkman triple system*, and denoted KTS(v). It is well known that a Steiner triple system on v points exists if and only if $v \equiv 1$ or $3 \pmod 6$ and that a KTS(v) exists if and only if $v \equiv 3 \pmod 6$. More generally, we call a 2-$(v, 3, \lambda)$ design a *triple system* and denote it by TS(v, λ). We refer the reader to [16, 17] for further results on triple systems. We have the following existence result.

Theorem 1.1 ([21]). *A TS(v, λ) exists if and only if $v \neq 2$ and $\lambda \equiv 0 \pmod{\gcd(v-2, 6)}$.*

A connection between zero sum flows on Steiner triple systems and more traditional geometric flows may be obtained by considering the well known embedding of the points of a TS$(7, 2)$ into the torus, so that the blocks form

equilateral triangles which may be coloured black and white, in such a way that no triangles of the same colour share an edge [19]. Giving each black triangle weight 1 and each white triangle weight -1, it is easy to see that this is equivalent to a zero-sum 2-flow of the $\mathrm{TS}(7, 2)$.

In this paper we consider the application of zero-sum flows to triple systems. In [3] it was conjectured that every $\mathrm{STS}(v)$ with $v > 7$ admits a zero-sum 3-flow. We generalise this conjecture to the case of arbitrary λ as follows.

Conjecture 1.2. *Every* $TS(v, \lambda)$ *admits a zero-sum* 3*-flow, except when* $(v, \lambda) \in \{(3, 1), (4, 2), (6, 2), (7, 1)\}$.

In support of the original conjecture, a computer search has shown that every $\mathrm{STS}(v)$, $7 < v \leq 15$, admits a zero-sum 3-flow; see [3]. In this paper, as further evidence in support of both the original conjecture in [3] and Conjecture 1.2, we show that for every admissible order $v > 7$ there exists an $\mathrm{STS}(v)$ which admits a zero-sum 3-flow. Further, we show in support of Conjecture 1.2 that for every admissible $(v, \lambda) \notin \{(3, 1), (4, 2), (6, 2), (7, 1)\}$, there exists a $\mathrm{TS}(v, \lambda)$ that admits a zero-sum 3-flow, except possibly when $\lambda = 2$ and $v \equiv 10 \pmod{12}$. In this latter case we show that a $\mathrm{TS}(v, \lambda)$ admitting a zero-sum 5-flow exists.

In the rest of this section we give some general results which apply to 2-designs in general. In the next section we then show the existence result for triple systems mentioned above. In Section 3 we give an $O((\lambda v)^2)$ bound for a zero-sum flow on a $\mathrm{TS}(v, \lambda)$. Finally, in Section 4, we provide some recursive $(2v + 1)$-constructions.

We now develop some general tools for determining zero-sum flows applicable to 2-designs. We begin with the following result on α-resolvable designs, recalling that we denote the number of α-resolution classes by $\rho = r/\alpha$.

Lemma 1.3. *An* α*-resolvable* 2-(v, k, λ) *design has a zero-sum* 3*-flow if* $\rho > 1$ *is odd and a zero-sum* 2*-flow if* ρ *is even.*

Proof. If ρ is even, we colour all of the blocks of each of the α-resolution classes alternately $+1$ and -1. Each pair of oppositely signed classes then generates a zero-sum flow. If ρ is odd we choose three classes and colour the blocks of two of them with $+1$ and the blocks of the third -2. The remaining classes (of which there are an even number) have their blocks coloured with $+1$ and -1 as above. $\square$

Taking $\alpha = 1$ we obtain the following result.

Theorem 1.4. *A resolvable design with at least two classes has a zero-sum 3-flow.*

Clearly there can be no resolvable STS$(6v+1)$. A *Hanani triple system* of order $6v+1$ has $3v$ almost parallel classes $P_1, P_2, \ldots, P_{3v}$ (each of which contain $6v$ points), and one partial parallel class P_0 of size v, containing $3v$ points. Thus, Hanani triple systems are in some sense as close to resolvable as one can get when the number of points is $1 \pmod 6$. There exists a Hanani triple system of order $6v+1$ if and only if $v \neq 1, 2$ [28]. We now show that these also admit a zero-sum 3-flow.

Theorem 1.5. *Let $v \geq 3$ be an integer. Any Hanani triple system of order $6v+1$ admits a zero-sum 3-flow.*

Proof. Let the blocks of P_0 be $B_1, B_2, \ldots, B_v$. For $i = 1, 2, \ldots, 3v$, let x_i be the point which does not appear in a block of P_i. Then the set of vertices which appear in blocks in P_0 is $\{x_1, x_2, \ldots, x_{3v}\}$.

First suppose that v is even. For $i = 1, 2, \ldots, v$, assign B_i the label $(-1)^i$. Next we colour the blocks of the remaining parallel classes. For each $i = 1, 2, \ldots, 3v$, if x_i is in a block with label j, assign label j to the blocks of parallel class P_i. It is easy to check that the resulting labelling is a zero-sum flow.

Now, suppose that v is odd. For $i = 1, 2, \ldots, v-3$, assign $(-1)^i$ to block B_i. Assign -1 to B_{v-2} and B_{v-1} and 2 to B_v. Again, for each $i = 1, 2, \ldots, 3v$, if x_i is in a block with label j, assign label j to parallel class P_i. □

Given a cyclic 2-(v, k, λ) design, we note that the set of blocks developed from a single starter which generates a full orbit forms a k-resolution class. Thus, if a cyclic design has no short orbits the design is k-resolvable. This observation, in conjunction with Theorem 1.4, gives us the following result.

Theorem 1.6. *A cyclic 2-(v, k, λ) design with no short orbits has a zero-sum 3-flow.*

Lemma II 2.3 of [8] states that the incidence matrix of a non-trivial symmetric design is non-singular. We thus have the following result.

Theorem 1.7. *A symmetric 2-design has no zero-sum flow.*

2 Existence of zero-sum flows for triple systems

In this section we show that for every pair (v, λ) such that a $\mathrm{TS}(v, \lambda)$ exists, there is one with a zero-sum 3-flow, except when $(v, \lambda) \in \{(3,1),(4,2),(6,2),(7,1)\}$ and except possibly when $v \equiv 10 \pmod{12}$ and $\lambda = 2$. Necessary and sufficient conditions for the existence of a $\mathrm{TS}(v, \lambda)$ were settled by Hanani [21] in 1961.

Theorem 2.1 ([21])**.** *The necessary conditions for the existence of a TS(v, λ) are:*

1. $\lambda \equiv 1, 5 \pmod 6$ *and* $v \equiv 1, 3 \pmod 6$*;*
2. $\lambda \equiv 2, 4 \pmod 6$ *and* $v \equiv 0, 1 \pmod 3$*;*
3. $\lambda \equiv 3 \pmod 6$ *and* $v \equiv 1 \pmod 2$*;*
4. $\lambda \equiv 0 \pmod 6$ *and* $v \geq 3$*;*

The (v, λ) values in Theorem 2.1 are called *admissible*.

We first give a non-existence result as a direct corollary of Theorem 1.7.

Lemma 2.2. *A zero-sum flow does not exist for TS$(3,1)$, TS$(4,2)$ and TS$(7,1)$.*

Next we give zero-sum flows for some small parameter values.

Lemma 2.3.

1. *There is no TS$(6,2)$ admitting a zero-sum 3-flow, but there is one with a zero-sum 4-flow.*
2. *There exists a TS$(6,4)$ with a zero-sum 2-flow*
3. *There exists a TS$(6,6)$ with a zero-sum 3-flow.*

Proof.

1. A computer search showed that the unique $\mathrm{TS}(6,2)$ has no zero-sum 3-flow. A zero-sum 4-flow is shown below.

B	123	134	145	156	126	235	346	245	356	246
$f(B)$	1	-2	2	-3	2	-2	1	1	2	-2

2. We create a TS$(6,4)$ with point set $\mathbb{Z}_5 \cup \{\infty\}$ by developing the following blocks $\pmod 5$, where ∞ is a fixed point. A zero-sum 2-flow is obtained by applying the given value of f to each block developed from that starter block.

$$\begin{array}{rcllll} B & = & (\infty,0,1) & (\infty,0,2) & (0,1,2) & (0,2,4) \\ f(B) & = & 1 & -1 & 1 & -1 \end{array}$$

3. Similarly, we create a TS$(6,6)$ with point set $\mathbb{Z}_5\cup\{\infty\}$ by developing the following blocks $\pmod 5$, where ∞ is a fixed point. A zero-sum 3-flow is obtained by applying the given value of f to each block developed from that starter block.

$$\begin{array}{rcllllll} B & = & (\infty,0,1) & (\infty,0,1) & (\infty,0,2) & (0,1,2) & (0,2,4) & (0,2,4) \\ f(B) & = & 2 & -1 & -1 & 2 & -1 & -1 \end{array}$$

□

The existence of resolvable triple systems has been determined (see [16]), giving us the following result.

Theorem 2.4. *A TS(v,λ) with a zero-sum 3-flow exists whenever $v \equiv 3 \pmod 6$, or $v \equiv 0 \pmod 6$ and λ is even, except when $(v,\lambda) \in \{(3,1),(6,2)\}$.*

Proof. A resolvable TS(v,λ) exists if and only if $v \equiv 3 \pmod 6$, or $v \equiv 0 \pmod 6$ and λ is even, $v \neq 6$, [16, 17]. The result then follows from Theorem 1.4, noting that when $v = 3$, Theorem 1.4 does not apply as there is only one resolution class. □

Cyclic triple systems have also been well studied; their existence was established in [15]. In [25] the structure of the short orbits is considered.

Theorem 2.5. *There exists a TS(v,λ) which admits a zero-sum 3-flow for the following:*

1. $\lambda \equiv 1 \pmod 6$ *and* $v \equiv 1 \pmod 6$*;*

2. $\lambda \equiv 2,10 \pmod{12}$ *and* $v \equiv 1,4,7 \pmod{12}$*;*

3. $\lambda \equiv 3 \pmod 6$ *and* $v \equiv 1 \pmod 2$*;*

4. $\lambda \equiv 4,8 \pmod{12}$ *and* $v \equiv 1 \pmod 3$*;*

5. $\lambda \equiv 6 \pmod{12}$ *and* $v \equiv 0, 1, 3 \pmod{4}$;

6. $\lambda \equiv 0 \pmod{12}$ *and* $v \geq 3$;

Proof. For the given values of v and λ, there exists a cyclic TS(v, λ) with no short orbits [15, 25]. Apply Theorem 1.6 to get the result. □

Lemma 2.6. *There exists a TS(v, λ) with a zero-sum 3-flow for every $\lambda \equiv 0 \pmod{6}$.*

Proof. A TS$(6, 6)$ with a zero-sum 3-flow is given in Lemma 2.3. For $v > 6$, there exists an idempotent Latin square L of order v with an orthogonal mate L'. We take the symbol sets of these squares to be $\mathbb{Z}_v$. We form the design by taking the $v(v-1)$ triples

$$\mathcal{B} = \{(i, j, L(i, j)) \mid i, j \in \mathbb{Z}_v,\ i \neq j\}.$$

The fact that L has an orthogonal mate, L', means that L can be decomposed into transversals, T_i, $i \in \mathbb{Z}_v$, each of which corresponds to the occurences of a given entry in L' (see [16]). We note that by reordering rows and columns and permuting symbols if necessary, we can arrange L so that T_{v-1} consists of the diagonal entries of L. Each T_i, $i \neq v-1$, then corresponds to a collection of blocks in the design. Taking the transversals in pairs, T_i, T_{i+1}, we label the blocks which come from T_i, $+1$, and T_{i+1}, -1; if v is even (so there is an odd number of transversals), we label the blocks from T_0, T_1, T_2 with $+2$, -1, -1 respectively. □

Theorem 2.7. *There exists a TS(v, λ) with a zero-sum 3-flow for every admissible pair (v, λ) except for $(v, \lambda) \in \{(3, 1), (4, 2), (6, 2), (7, 1)\}$ and except possibly when $v \equiv 10 \pmod{12}$ and $\lambda = 2$.*

Proof. The non-existence of any zero-sum flow for a TS$(3, 1)$, TS$(4, 2)$ and TS$(7, 1)$ are given in Lemma 2.2 and the non-existence of a zero-sum 3-flow for a TS$(6, 2)$ is in Lemma 2.3. We note that for a fixed v we may take copies of designs to build up λ. In particular, the existence of a TS$(v, 6)$ from Lemma 2.3 means that often it suffices to consider only the first value (mod 6) in λ. We consider the cases in $\lambda \pmod{6}$.

$\boldsymbol{\lambda \equiv 1, 5 \pmod{6}}$ ($\boldsymbol{v \equiv 1, 3 \pmod{6}}$)

For $v \equiv 1 \pmod{6}$ the result follows from Theorem 2.5. For $v \equiv 3 \pmod{6}$ the result follows from Theorem 2.4.

$\boldsymbol{\lambda \equiv 2, 4 \pmod 6}$ ($\boldsymbol{v \equiv 0, 1 \pmod 3}$)

When $v \equiv 0 \pmod 3$, $(v, \lambda) \neq (6, 2)$, the result follows from Theorem 2.4.

When $v \equiv 1 \pmod 3$ this is covered by Theorem 2.5, except when $v \equiv 10 \pmod{12}$ and $\lambda \equiv 2, 10 \pmod{12}$. We now suppose $v \equiv 10 \pmod{12}$ and consider the cases in λ.

When $\lambda \equiv 10 \pmod{12}$, we may combine the blocks of a TS$(v, 4)$ design with a zero-sum 3-flow and a TS$(v, \lambda - 4)$ with a zero-sum 3-flow. Note that $\lambda - 4 \equiv 6 \pmod{12}$, so this design exists by Lemma 2.6.

When $\lambda \equiv 2 \pmod{12}$, $\lambda > 2$, we may combine the blocks of a TS$(v, 4)$ with a zero-sum 3-flow and a TS$(v, \lambda - 4)$ design with a zero-sum 3-flow. Note that $\lambda - 4 \equiv 10 \pmod{12}$, so this is the design above.

$\boldsymbol{\lambda \equiv 3 \pmod 6}$ ($\boldsymbol{v \equiv 1 \pmod 2}$)

This is covered by Theorem 2.5.

$\boldsymbol{\lambda \equiv 0 \pmod 6}$ ($\boldsymbol{v \geq 3}$)

This is covered by Lemma 2.6.

□

Finally we give the following theorem which shows that in the case of $v \equiv 10 \pmod{12}$ and $\lambda = 2$, there is a TS(v, λ) with a zero-sum 5-flow.

Theorem 2.8. *There exists a TS(v, λ) with a zero-sum 5-flow for every $v \equiv 4 \pmod 6$, $v \neq 4$ and $\lambda \equiv 2, 4 \pmod 6$. There exists a TS$(v, \lambda)$ with a zero-sum 3-flow for every $v \equiv 1 \pmod 6$, $v \neq 7, 19$ and $\lambda \equiv 2, 4 \pmod 6$.*

Proof. We use a modification of the Bose construction; see [17]. We construct the design on $(\mathbb{Z}_n \times \mathbb{Z}_3) \cup \{\infty\}$, where $n = (v-1)/3$, and denote points by x_i, where $x \in \mathbb{Z}_n$ and $i \in \mathbb{Z}_3$. For $n \neq 2, 6$ there exists an idempotent Latin square L of side n with an orthogonal mate L' [16]. For each pair $x, y \in \mathbb{Z}_n$, $x \neq y$, we form the design by taking the triples $(x_i, y_i, L(x, y)_{i+1})$. We label the blocks which come from the transversals of L, T_i, $i < v - 1$, with $+1$, -1 in pairs; if v is odd, we label the blocks from T_0, T_1, T_2 with $+2, -1, -1$ respectively. This creates a partial design with a zero-sum 3-flow on the points of $\mathbb{Z}_n \times \mathbb{Z}_3$.

For each vertical, $B_x = (x_0, x_1, x_2)$, $x \in \mathbb{Z}_n$, we place a TS$(4,2)$ on $B_x \cup \{\infty\}$ with the flow values as indicated below.

$$\begin{array}{rcccc} B = & (\infty, x_0, x_1) & (\infty, x_1, x_2) & (\infty, x_0, x_2) & (x_0, x_1, x_2) \\ f_\alpha(B) = & \alpha & \alpha & \alpha & -2\alpha \end{array}$$

This gives weight 0 to each $x_i \in \mathbb{Z}_n \times \mathbb{Z}_3$ and weight 3α to ∞. If $n = \frac{v-1}{3}$ is even we label the designs on the verticals with $\alpha = +1, -1$ in pairs, to get a zero-sum 3-flow. If n is odd, we label three designs on the verticals with $\alpha = 1, 1, -2$ (which gives a block of weight 4) and the rest with $\alpha = +1, -1$ in pairs to get a zero-sum 5-flow ($n > 2$). □

3 An $O((\lambda v)^2)$ Bound for Zero-Sum Flows

In this section we establish an $O((\lambda v)^2)$ bound on the size of a zero-sum flow on a TS(v, λ) when $v > 4$, $v \neq 7$. Further, this zero-sum $O((\lambda v)^2)$-flow takes at most five distinct values.

Theorem 3.1. *For $v > 4$, $v \neq 7$, every TS(v, λ) admits a zero-sum flow whose entries are in the set*

$$S = \left\{ \frac{-\lambda(v-3)}{2}\left(\frac{\lambda(v-3)}{2} - 2\right),\ -3\lambda,\ \frac{3\lambda(v-3)}{2},\ \frac{\lambda(v-7)}{2},\ \lambda(v-4) \right\}.$$

Proof. Consider a TS(v, λ) with incidence matrix N. We know that the number of appearances of each element in blocks is $r = \frac{\lambda(v-1)}{2}$. We show that the last column of N is a linear combination of the other columns with coefficients from the set $3S$.

By a suitable ordering of the elements of $\{1, \ldots, v\}$, we can assume that the last column of N is $Z = [1, 1, 1, 0, \ldots, 0]^T$. Now, we remove the last column of N and call the remaining matrix M. We then have the following equality:

$$L = MM^T = \begin{bmatrix} (r-\lambda)I_3 + (\lambda - 1)J_3 & \lambda J_{3,v-3} \\ \lambda J_{v-3,3} & (r-\lambda)I_{v-3} + \lambda J_{v-3} \end{bmatrix},$$

where $J_{p,q}$ is the $p \times q$ all-1 matrix. For simplicity we denote $J_{p,p}$ by J_p.

We now show that the system of linear equations $LY = tZ$ has integral solutions such that each component of Y is in the set $\{-3\lambda, \frac{3\lambda(v-3)}{2}\}$. Assume

that $Y = [a, a, a, b, \dots, b]^T$, where a and b are unknown variables and there are $v-3$ bs. We may solve the equations

$$(r-\lambda)a + 2(\lambda-1)a + \lambda(v-3)b = t \quad \text{and} \quad 3\lambda a + (r + \lambda(v-4))b = 0,$$

to get $a = r - \lambda(v-4) = \frac{3\lambda(v-3)}{2}$, $b = -3\lambda$ $(v > 4)$ and

$$t = (r+\lambda-2)(r-\lambda(v-4)) - 3\lambda^2(v-3) = \frac{-3\lambda(v-3)}{2}\left(\frac{\lambda(v-3)}{2} - 2\right).$$

Now, since $MM^TY = tZ$ and each row of M^T has exactly three 1s, we may conclude that there exists a vector, X, in the nullspace of N whose components are all of the form $-t$, $3a$, $2a+b$, $a+2b$ or $3b$. Finally, we note that when $v > 4$, $v \neq 7$, all of the components of X are non-zero integers which are divisible by 3. Now, $X' = \frac{1}{3}X$ is also in the nullspace of N and all of its components are in the set S. □

4 $(2v+1)$-construction for zero-sum flows on STS(v)

In this section we show that the standard $(2v+1)$-construction for Steiner triple systems can be adapted to respect zero-sum flows in many cases. While we work in the case $\lambda = 1$, the generalisation to higher λ is clear.

We say that a graph G has a *k-null 1-factorisation* if G has a zero-sum k-flow and there is a 1-factorisation in which the weight of each 1-factor is zero. We call each 1-factor of a k-null 1-factorisation a *k-null 1-factor*.

Lemma 4.1. *If K_{v+1} has a k-null 1-factorisation and there exists an STS(v), $\mathcal{S}$, with a zero-sum ℓ-flow, then $\mathcal{S}$ can be embedded into an STS$(2v+1)$ with a zero-sum* $\max(k, \ell)$*-flow.*

Proof. Let $(X, \mathcal{B})$ be an STS(v), with $X = \{x_1, \dots, x_v\}$, and let Y be a set of order $v+1$. We will construct the new design on $X \cup Y$. Construct a k-null 1-factorisation on K_{v+1} with point set Y. Let $F_1, \dots, F_v$ be the 1-factors in this 1-factorisation, and suppose that $F_i = \{\{y_{ij}, z_{ij}\} \mid 1 \leq j \leq \frac{v+1}{2}\}$, for $i = 1, \dots, v$. Form the triples

$$\mathcal{C} = \bigcup_{1\leq i\leq v,\, 1\leq j\leq \frac{v+1}{2}} \{x_i, y_{ij}, z_{ij}\},$$

it is easy to see that $\mathcal{B} \cup \mathcal{C}$ is an STS$(2v+1)$. In order to obtain a zero-sum $\max(k,\ell)$-flow, we retain the original weighting on $\mathcal{B}$, and for each triple $\{x_i, y_{ij}, z_{ij}\} \in \mathcal{C}$, we give it the weight of the edge $\{y_{ij}, z_{ij}\}$ in F_i. It is not hard to see that we obtain a zero-sum $\max(k,\ell)$-flow on the STS$(2v+1)$, $(X \cup Y, \mathcal{B} \cup \mathcal{C})$. □

Lemma 4.2. *There exists a 3-null 1-factorisation of $K_{n,n}$ for every $n \geq 3$.*

Proof. First assume that $n \neq 6$ and let L_1 and L_2 be two orthogonal Latin squares of order n. Let $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots, v_n\}$ be the two parts of the complete bipartite graph $K_{n,n}$. We use L_1 to define a colouring, c, on the edges of $K_{n,n}$, and L_2 to factor $K_{n,n}$ into k-null 1-factors as follows. For each edge $u_iv_j \in E(K_{n,n})$, $1 \leq i,j \leq n$, we give u_iv_j colour $c(u_i,v_j) = L_1(i,j)$. We now define a 1-factorisation of $K_{n,n}$ by, $F_k = \{(u_i,v_j) \mid L_2(i,j) = k\}$. Note that since L_1 and L_2 are orthogonal, each F_i contains exactly one edge of each colour. If n is even, then we assign 1 to every edge whose colour is in the set $\{1, \dots, \frac{n}{2}\}$ and assign -1 to the remaining edges. If n is odd, then we assign $2, -1, -1$ to all edges with colours, $1, 2$ and 3, respectively and assign $1, -1$ to all edges with colours $4, 5, \dots, n$, alternately. It is not hard to see that this assignment is the desired zero-sum 3-flow for $K_{n,n}$.

For the case $n = 6$, the edges of $K_{6,6}$ can be decomposed into four subgraphs isomorphic to $K_{3,3}$. Since every $K_{3,3}$ has the desired zero-sum 3-flow, we are done. □

Lemma 4.3. *There exists a 3-null 1-factorisation of K_n for every $n \equiv 0 \pmod 4$, $n \neq 4$.*

Proof. Let $n = 4r$, $r > 1$, and consider the complete graph K_n as the join of two complete graphs K_{2r} and K_{2r}. By Lemma 4.2, $K_{2r,2r}$ has a 3-null 1-factorisation. Let $M_1, \dots, M_{2r-1}$ and $M'_1, \dots, M'_{2r-1}$ be two 1-factorisations for the first and the second K_{2r}, respectively. Then $\{M_i \cup M'_i \mid 1 \leq i \leq 2r-1\}$ forms a 1-factorisation for the disjoint union of the two K_{2r}. Now assign 2 to all edges of M_1, -2 to all edges of M'_1, -1 to all edges of M_2 and 1 to all edges of M'_2. For each i, $3 \leq i \leq 2r-1$ assign -1 and 1 to all edges of M_i, alternately. For each i, $3 \leq i \leq 2r-1$ assign 1 and -1 to M'_i, alternately. □

Now, using Lemmas 4.1 and 4.3, we have the following result.

Theorem 4.4. *If $v \equiv 3 \pmod 4$, $v > 3$, then every* STS(v) *with a zero-sum 3-flow can be embedded into an* STS$(2v+1)$ *with a zero-sum 3-flow.*

We note that the requirement $v \equiv 3 \pmod 4$ and the existence of an STS(v) implies that $v \equiv 3, 7 \pmod{12}$.

A similar proof to that of Lemma 4.3 shows that if $n \equiv 0 \pmod 8$ then K_n has a zero-sum 2-flow with a 1-factorisation in which the weight of each 1-factor is zero. By Lemma 4.1, this shows that when $v \equiv 7 \pmod 8$ every STS(v) with a zero-sum 2-flow can be embedded into an STS$(2v+1)$ with a zero-sum 2-flow.

Lemma 4.5. *There exists a* 4*-null 1-factorisation of* K_{6k+4} *for every* $k > 1$.

Proof. Let the point set of K_{6k+4} be $\mathbb{Z}_{6k+3} \cup \{\infty\}$ and let $F_1, \ldots, F_{6k+3}$ be the 1-factorisation of K_{6k+4} defined by $F_i = F_1 + i$, $0 \le i \le 6k+2$, where $F_1 = \{(x, -x) \mid x \in \{1, \ldots, 3k+1\}\} \cup \{(0, \infty)\}$. We claim that K_{6k+4} can be partitioned into $2k+1$ cubic graphs each isomorphic to

$$K_{3,3} \cup K_{3,3} \cup \cdots \cup K_{3,3} \cup K_4,$$

where the number of $K_{3,3}$ is k.

It is not hard to see that $F_1 \cup F_{2k+2} \cup F_{4k+3}$ is a disjoint union of a K_4 with vertex set $\{\pm(2k+1), 0, \infty\}$ and k copies of $K_{3,3}$ with the vertex sets

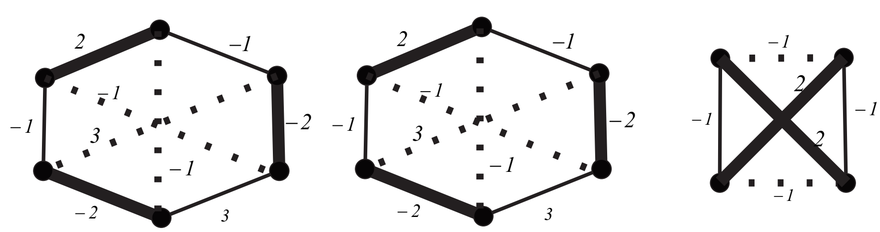
$$\{\pm i, \pm(2k+1-i), \pm(2k+1+i)\},\ 1 \le i \le k,$$

and partite sets

$$X_i = \{i, -(2k+1-i), 2k+1+i\} \text{ and } Y_i = \{-i, 2k+1-i, -(2k+1+i)\}.$$

Clearly, $F_i \cup F_{2k+1+i} \cup F_{4k+2+i} \simeq F_1 \cup F_{2k+2} \cup F_{4k+3}$, for $i = 1, \ldots, 2k+1$ and so it can also be decomposed as required.

Now, consider the following edge assignment for disjoint union of two $K_{3,3}$ and a K_4.



Thus, by this assignment, $K_{3,3} \cup K_{3,3} \cup K_4$ admits a 4-null 1-factorisation. On the other hand, by Lemma 4.2, $K_{3,3}$ admits a 4-null 1-factorisation. $\square$

We may use Lemma 4.5 above in Lemma 4.1 to get the following result.

Theorem 4.6. *If* $v \equiv 3 \pmod 6$*,* $v > 9$*, then every* $\mathrm{STS}(v)$ *with a zero-sum* 4*-flow can be embedded in an* $\mathrm{STS}(2v+1)$ *with a zero-sum* 4*-flow.*

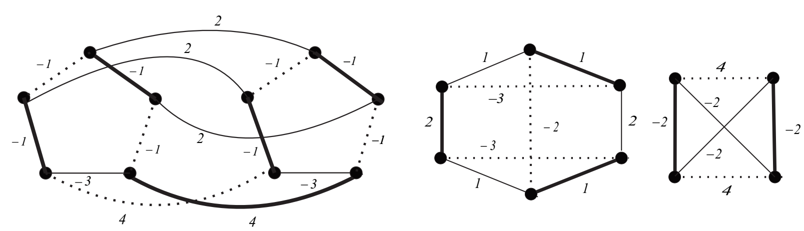
Given two graphs G and H, the *Cartesian product* of G and H, $G \Box H$, is the graph with vertex set $V(G) \times V(H)$, and $(x,a)\,(y,b) \in E(G \Box H)$ if and only if either $x = y$ and $a\,b \in E(H)$, or $a = b$ and $x\,y \in E(G)$. We can deal with the case $v = 9$ allowing for a 5-flow.

Lemma 4.7. *There exists a* 5*-null 1-factorisation of* K_{10}*.*

Proof. It is not hard to see that K_{10} can be decomposed into 3 cubic graphs

$$C_5 \Box K_2, C_5 \Box K_2, (K_3 \Box K_2) \cup K_4.$$

Now, the following edge assignments imply that there is a 5-null 1-factorisation of K_{10}.



$\square$

Given two graphs G and H, the *wreath product* of G and H, $G \wr H$, is the graph with vertex set $V(G) \times V(H)$, and $(x,a)\,(y,b) \in E(G \wr H)$ if and only if $x\,y \in E(G)$, or $x = y$ and $a\,b \in E(H)$.

A 2-*factor* is a collection of cycles that spans all vertices of the graph. A 2-*factorisation* of a graph G is an edge decomposition of G into 2-factors. The problem of finding an F-factorisation of K_n, where F is a given 2-factor of order n, is the well known Oberwolfach problem. Clearly a 2-factorisation of K_n cannot exist when n is even; in this case it is common practice to consider factorisations of $K_n - I$, the complete graph with the edges of a 1-factor I removed. It is known that the Oberwolfach problem has no solution when $F \in \{C_3 \cup C_3, C_3 \cup C_3 \cup C_3 \cup C_3, C_4 \cup C_5, C_3 \cup C_3 \cup C_5\}$. Otherwise, a solution is known for every case where $n \leq 40$ [18], and if every component of the factor is isomorphic [6, 7, 23]. The case where F is bipartite is nearly

solved [10, 20]; the case where F consists of exactly two parts is solved [9, 26]. Rotational solutions have been studied [12, 13, 14] and many other families are known [11, 22, 24], but no general solution is known. See [16, Section VI.12] for a survey. Häggkvist proved the following very useful result in [20].

Lemma 4.8 ([20]). *For any $m > 1$ and for each bipartite 2-regular graph F of order $2m$, there exists a 2-factorisation of $C_m \wr \overline{K_2}$, in which each 2-factor is isomorphic to F.*

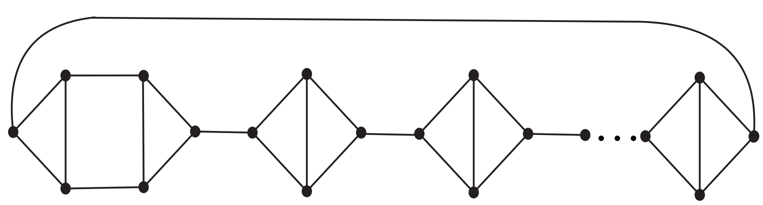
Since it is well known that K_{2k+1} has a Hamiltonian factorisation, we obtain the following theorem, which was proved in [6].

Theorem 4.9 ([6]). *For every positive integer k, the graph K_{4k+2} can be decomposed into $k-1$, graphs isomorphic to $C_{2k+1} \wr \overline{K_2}$, and one graph isomorphic to $C_{2k+1} \wr K_2$.*

Lemma 4.10. *There exists a 5-null 1-factorisation of K_{4k+2} for every $k > 1$.*

Proof. By Theorem 4.9, K_{4k+2} can be decomposed into $k-1$, graphs isomorphic to $C_{2k+1} \wr \overline{K_2}$, and one $C_{2k+1} \wr K_2$. Now, let F be a 2-regular graph which is disjoint union of one C_6 and $k-1$, C_4s. By Lemma 4.8, $C_{2k+1} \wr \overline{K_2}$, has a

Figure 1: The cubic graph H_i



2-factorisation in which each 2-factor is isomorphic to F. Assign -2 and 2 to the edges of C_6, alternately. If the number of C_4 in F is even, then assign -2 and 2 to the edges of one C_4, alternately and assign -1 and 1 to the edges of each other C_4, alternately. It is not hard to see that F has a 3-null 1-factorisation. If the number of C_4 is odd, then assign -3 and 3 to the edges of one C_4, alternately and assign -1 and 1 to the edges of each other C_4, alternately. Again, it is not hard to see that F has a 3-null 1-factorisation.

Let H_i be the cubic graph shown in Figure 1, where there are exactly i blocks of K_4 minus one edge. Now, consider the 5-regular graph $C_{2k+1} \wr K_2$,

Figure 2: The edge decomposition of $C_{2k+1} \wr K_2$ into F and H_{k-1}

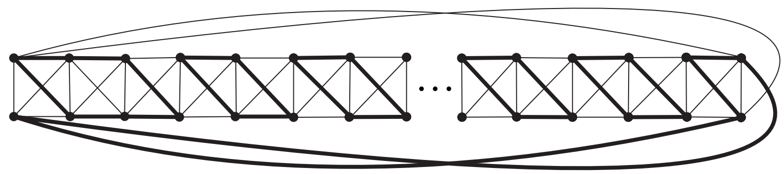


Figure 3: 5-flow on H_1

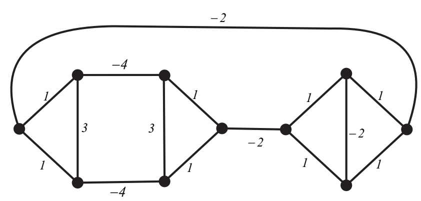


Figure 2 shows that $C_{2k+1} \wr K_2$ can be decomposed into a copy of F and the cubic graph H_{k-1}. Figures 3, 4 and 5 show a 5-null 1-facorisation of H_1, H_3 and H_5, respectively. Figure 6 gives a graph which can be appended to H_1, H_3 or H_5 to get a zero-sum 5-flow with a 1-factor in which the weight of each 1-factor is zero for any H_i, except $i = 2$.

Now, for the case $i = 2$, so $k = 3$, let L be the bipartite 2-factor which is the disjoint union of a C_6 and a C_8. It is not hard to see that $C_7 \wr K_2$ can be decomposed into L and the cubic graph given in Figure 7, which has a zero-sum 5-flow with a desired 1-factor.

Now, alternately give weight 3 and -3 to all edges of C_8 and 4 and -4 to all edges of C_6 in L, alternately. This implies that K_{14} has a zero-sum 5-flow

Figure 4: 5-flow on H_3

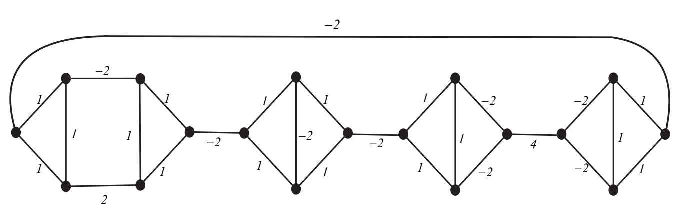


Figure 5: 5-flow on H_5

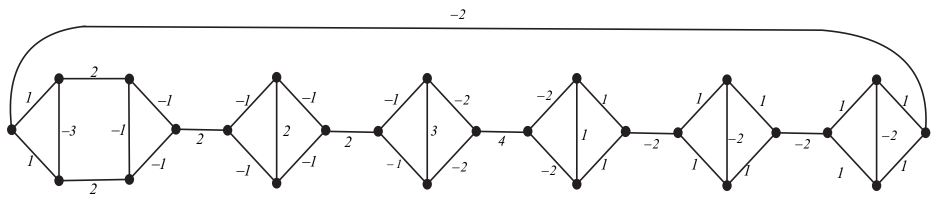


Figure 6: Continuation Graph

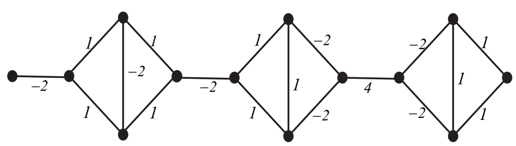
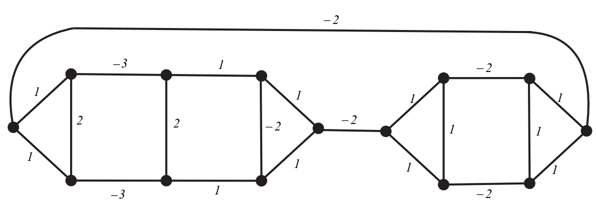


Figure 7: Zero-sum 5-flow on L



with a 1-factorisation in which the weight of each 1-factor is zero. $\square$

Now, using Lemma 4.10 in Lemma 4.1 we get the following result.

Theorem 4.11. *If $v \equiv 1 \pmod 4$, then every $STS(v)$ with a zero-sum 5-flow can be embedded in an $STS(2v+1)$ with a zero-sum 5-flow.*

Noting that the necessary conditions for the existence of an STS(v) are $v \equiv 1, 3, 7, 9 \pmod{12}$, we may summarise the results of this section in the following theorem.

Theorem 4.12. *An $STS(v)$ admitting a zero-sum k-flow may be embedded in an $STS(2v+1)$ which admits a zero-sum k-flow under the following conditions:*

- $k \geq 3$, $v \equiv 3, 7 \pmod{12}$, $v > 7$;
- $k \geq 4$, $v \equiv 9 \pmod{12}$, $v > 9$;
- $k \geq 5$, $v \equiv 1 \pmod{12}$, *or* $v = 9$.

We note that the zero-sum flow on the blocks of the embedded STS(v) remains unchanged.

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